A New Iterative WLS Chebyshev Approximation Method for the Design of Two-Dimensional FIR Digital Filters

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Abstract—In this paper, based on Chi-Chiou's weighted least-squares (WLS) Chebyshev approximation method which is for the design of one-dimensional (1-D) FIR digital filters with arbitrary complex frequency response, we propose a new iterative WLS Chebyshev approximation method for the design of two-dimensional (2-D) FIR digital filters with arbitrary complex frequency response. Several design examples are provided to justify the good performance of the proposed approximation method.

I. Introduction

Assume that the desired frequency response $H_d(\omega_1, \omega_2)$ is defined over p disjoint nontransition bands $R_1, R_2, \ldots, R_p$. Let

$$R = R_1 \cup R_2 \cup \ldots R_p$$

which includes all nontransition bands. The approximation error is defined as

$$E(\omega_1,\omega_2) = H_d(\omega_1,\omega_2) - H(\omega_1,\omega_2), \ \ \forall(\omega_1,\omega_2) \in R \quad (1)$$

in which

$$H(\omega_1,\omega_2) = \sum\sum_{(n_1,n_2)\in S} h(n_1,n_2)e^{-j(\omega_1 n_1+\omega_2 n_2)} \quad (2)$$

where S is the domain of support of the two-dimensional (2-D) finite impulse response (FIR) filter under design. It is well known that the Chebyshev (minimax) approximation is to find a set of optimum filter coefficients $\{h(n_1,n_2)|(n_1,n_2) \in S\}$ such that the objective function

$$J_C(h) = max\{|E'(\omega_1,\omega_2)|, (\omega_1,\omega_2) \in R\} \quad (3)$$

is minimum, where

$$E'(\omega_1,\omega_2) = W_e(\omega_1,\omega_2) \cdot E(\omega_1,\omega_2) \quad (4)$$

is the weighted approximation error and $W_e(\omega_1,\omega_2)$ is a prescribed weighting function. Chebyshev (minimax) criterion based filters are basically quasi-equiripple and can only be obtained by approximation methods such as exchange ascent algorithms [1-4], Charalambous' weighted least-squares (WLS) method [5] and Algazi, Suk and Rim's WLS method [6], but all of them are for the design of 2-D linear phase and zero-phase FIR filters.

Recently, Chi and Chiou [7] proposed a computationally efficient WLS Chebyshev approximation method for the design of 1-D FIR digital filters with arbitrary complex frequency response. Based on their 1-D WLS Chebyshev approximation method, they [8] proposed an iterative WLS Chebyshev approximation method for the design of 2-D FIR digital filters with arbitrary complex frequency response. In this paper, we further propose another WLS Chebyshev approximation method which not only performs better but also converges much faster than the one reported in [8].

II. WLS Estimation of 2-D FIR Filters

Assume that the desired tolerance error ratio among $R_1, R_2, \ldots, R_p$ is specified by

$$\delta_1 : \delta_2 : \ldots : \delta_p = \rho_1 : \rho_2 : \ldots : \rho_p$$

where

$$\delta_k = max\{|E(\omega_1,\omega_2)|, (\omega_1,\omega_2) \in R_k\}.$$

Let us consider M sampling points in R, denoted $(\omega_{1i},\omega_{2i})$, $i = 1,2,\ldots,M$. Let

$$\boldsymbol{H}_d = [H_d(\omega_{11},\omega_{21}), H_d(\omega_{12},\omega_{22}), \ldots, H_d(\omega_{1M},\omega_{2M})]^t,$$

$$\boldsymbol{E} = [E(\omega_{11},\omega_{21}), E(\omega_{12},\omega_{22}), \ldots, E(\omega_{1M},\omega_{2M})]^t.$$

Then we can concatenate the approximation errors $E(\omega_{1i},\omega_{2i})$, $i = 1,2,\ldots,M$ as the following complex linear vector model:

$$\boldsymbol{E} = \boldsymbol{H}_d - D \cdot \boldsymbol{h} \quad (5)$$

where $\boldsymbol{h}$ is a vector containing all nonredundant $h(n_1,n_2)'s$, D is an $M \times dim(\boldsymbol{h})$ complex matrix whose elements can be easily determined from the definition of $H(\omega_1,\omega_2)$. The sum of weighted error squares is defined as

$$J(\boldsymbol{h}) = \sum_{i=1}^{M} w_i|E(\omega_{1i},\omega_{2i})|^2 = \boldsymbol{E} \cdot W \cdot \boldsymbol{E}^H \quad (6)$$

where $\boldsymbol{E}^H$ is the complex conjugate transpose of $\boldsymbol{E}$ and $W = diag[w_1, w_2, \ldots, w_M]$ with $w_i \geq 0$ for all $1 \leq i \leq M$. It is well known that the WLS estimate, $\hat{\boldsymbol{h}}$, of $\boldsymbol{h}$ which minimizes $J(\boldsymbol{h})$ is given by

$$\hat{\boldsymbol{h}} = [D^H W D]^{-1} D^H W \boldsymbol{H}_d. \quad (7)$$

A well-known property of WLS estimators, on which the iterative WLS Chebyshev approximation method to be presented below is based, is as follows:

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(P1) *The larger the weight* w_i*, the smaller is the absolute error* $|E(\omega_{1i}, \omega_{2i})|$.

III. A New Iterative WLS Chebyshev Approximation Method

Let $a(j)$ denote the jth largest local maximum of $|E'(\omega_1, \omega_2)|$ for $(\omega_1, \omega_2) \in R$, where $E'(\omega_1, \omega_2)$ is given by (4) in which $W_e(\omega_1, \omega_2)$ is a piecewise-constant weighting function as

$$W_e(\omega_1, \omega_2) = 1/\rho_k, \ \ if \ \ (\omega_1, \omega_2) \in R_k, \tag{8}$$

and let

$$a_{ave} = \frac{1}{\lceil m/2 \rceil} \sum_{j=1}^{\lceil m/2 \rceil} a(j)$$

where m is the total number of weighted-error local maxima $a(j)$'s, and $\lceil m/2 \rceil = m/2$ if m is even and $\lceil m/2 \rceil = (m+1)/2$ if m is odd.

The new WLS Chebyshev approximation method below is an iterative algorithm based on (P1) for finding the optimum w_i such that $|E'(\omega_1, \omega_2)|$ is quasi-equiripple for $(\omega_1, \omega_2) \in R$.

NEW ITERATIVE APPROXIMATION ALGORITHM :

- Initial values of w_i:

$$w_i^{(0)} = W_e(\omega_{1i}, \omega_{2i}), \ \ i = 1, 2, \ldots, M. \tag{9}$$

- For the nth iteration:

(A) Compute the WLS estimate $\hat{h}$ computed by (7) with $w_i = w_i^{(n-1)}$.

(B) Compute the approximation error $\boldsymbol{E}$ by (5) with $h = \hat{h}$.

(C) Search for all weighted-error local maxima $a(j)$'s from $\{|E'(\omega_{1i}, \omega_{2i})|, \ i = 1, 2, \ldots, M\}$.

(D) If

$$(a(1) - a_{ave})/a_{ave} \le \alpha \tag{10}$$

where α is a preassigned small positive constant, we have obtained the desired $\hat{h}$, otherwise go to the $(n+1)$th iteration with the weighting function updated by

$$w_i^{(n)} = \frac{w_i^{(n-1)} |E'(\omega_{1i}, \omega_{2i})|^q}{max\left\{ w_i^{(n-1)} |E'(\omega_{1i}, \omega_{2i})|^q \right\}} + c \tag{11}$$

where q is a positive real number, c is a small positive real number, and both of them must be assigned in advance.

Next, let us briefly discuss how to select the parameters q and c, respectively.

The parameter q plays a role similar to the step size parameter in the well-known least-mean-square (LMS) adaptive filter. For a small q, the approximation error $|E'(\omega_1, \omega_2)|$ follows a smooth path, and for a large q, the approximation error $|E'(\omega_1, \omega_2)|$ exhibits oscillations from iteration to iteration. Empirically, we found that the proposed method converges for $1 \le q < 2$ and that $q = 1.5$ is a good choice. On the other hand, we limit the weight $w_i^{(n)}$ to $c \le w_i^{(n)} \le 1 + c$ $(c > 0)$ instead of $0 \le w_i^{(n)} \le 1$ to avoid possible zero weights due to accidental zero approximation errors at some frequencies which result in the approximation errors at those frequencies uncontrollable in all the subsequent iterations. As long as $(1+c)/c$ is large enough the minimum of $\max\{|E'(\omega_1, \omega_2)|\}$ can be attained.

One can see, from (10), that the convergence of the proposed approximation method is determined by whether the largest $\lceil m/2 \rceil$ weighted-error local maxima $\{a(1), a(2), \ldots, a(\lceil m/2 \rceil)\}$ of $|E'(\omega_1, \omega_2)|$ in R are uniform. By our experience, all the weighted-error local maxima $a(j)$ of the designed filter basically form two groups. One consists of larger weighted-error local maxima which are almost equal and the other consists of smaller unequal weighted-error local maxima. The number of $a(j)$'s in the former is always much larger than that in the latter, and the former always includes uniform weighted approximation errors $\delta_1/\rho_1 \cong \delta_2/\rho_2 \cong \cdots \cong \delta_p/\rho_p$ because $a(1) \cong a(2) \cong \cdots \cong a(\lceil m/2 \rceil)$ and $\lceil m/2 \rceil >> p$ in general. Therefore, the designed FIR filter is a nearly equiripple filter with the desired error approximation error ratio $\delta_1 : \delta_2 : \cdots : \delta_p = \rho_1 : \rho_2 : \cdots : \rho_p$.

Although the proposed approximation method was illuminated via the design of a complex FIR filter, it is surely applicable in the case of real filter coefficients by converting the complex linear vector model given by (5) into a real linear vector model which can be easily obtained by concatenating the real part and the imaginary part of $E(\omega_{1i}, \omega_{2i})$ as a $2M \times 1$ vector. The real WLS estimate $\hat{h}$ associated with the resultant real linear vector model has a form similar to (7) except that the matrix D here is a real $2M \times \dim(h)$ matrix whose components can be easily determined from the real part and the imaginary part of $H(\omega_1, \omega_2)$.

IV. Design Examples

In the following design examples, the sampling points $(\omega_{1i}, \omega_{2i}), i = 1, 2, \ldots, M$, include sampling points on the boundaries of transition bands and uniform sampling points within nontransition bands on the Cartesian grid $(k_1 \Delta\omega, k_2 \Delta\omega)$. The optimum filter was obtained by the proposed approximation method with the convergence parameter α (see (10)) set to 0.02.

Example 1: Zero-Phase Eightfold Symmetric Lowpass Filter

The desired frequency response is as follows:

$$H_d(\omega_1,\omega_2) = \begin{cases} 1, & \sqrt{\omega_1^2+\omega_2^2} \le 0.4\pi \\ 0, & \sqrt{\omega_1^2+\omega_2^2} \ge 0.6\pi \end{cases}$$

Assume that the desired tolerance error ratio is $\rho_1/\rho_2 = 1$ and that the FIR filter to be designed is a zero-phase eightfold symmetric $(2N+1) \times (2N+1)$-point (i.e., $S = \{(n_1,n_2)| -N \le n_1 \le N, -N \le n_2 \le N\}$) lowpass filter.

With the parameter q set to 1.5, c set to 10^{-4} and $\Delta\omega$ set to $\pi/32$ in the proposed approximation method, the numerical results of the designed filters by the proposed method as well as the corresponding results reported in [4] and [6] are shown in Table 1. We can see, from this table, that maximum errors associated with the proposed method are slightly larger than the corresponding maximum errors associated with Harris' steepest ascent method [4], while they are smaller than those associated with the modified Lawson's method [6]. Moreover, iterations spent by the proposed method are also fewer than those spent by the modified Lawson's algorithm.

Example 2: Real Nonlinear-Phase Lowpass Filter with Constant Group Delay

The desired frequency response is as follows:

$$H_d(\omega_1,\omega_2) = \begin{cases} e^{-j(\omega_1\tau_1+\omega_2\tau_2)}, & \sqrt{\omega_1^2+\omega_2^2} \le 0.4\pi \\ 0, & \sqrt{\omega_1^2+\omega_2^2} \ge 0.6\pi \end{cases}$$

where $\tau_1 = \tau_2 = 4$. The FIR filter to be designed is a 10×10-point real causal filter (i.e., $S = \{(n_1,n_2)|\ 0 \le n_1 \le 9,\ 0 \le n_2 \le 9\}$) is nonlinear-phase although it has constant group delay with respect to both ω_1 and ω_2. The desired tolerance error ratio is $\rho_1/\rho_2 = 1$.

In this example, c=10 , q=1.5 and $\Delta\omega = \pi/32$ were used. The designed magnitude response is shown in Figure 1 from which one can see that the designed filter is indeed quasi-equiripple. The number of iterations spent was 19. The maximum errors δ_1 in the passband and δ_2 in the stopband are 0.0826 and 0.0824, respectively. Note that $\delta_1/\delta_2 = 1.0022 \approx \rho_1/\rho_2 = 1$. Maximum deviations of both group delay with respect to ω_1 and that with respect to ω_2 in the passband are equal to 0.4044.

Example 3: Circular-Passband Complex Filter

Let

$Q(r) = \{(\omega_1,\omega_2)|(\omega_1 - 0.125\pi)^2 + (\omega_2 - 0.125\pi)^2 \le r^2\}$.

The desired frequency response is as follows:

$$H_d(\omega_1,\omega_2) = \begin{cases} e^{-j(\omega_1\tau_1+\omega_2\tau_2)}, & (\omega_1,\omega_2) \in Q(0.4\pi) \\ 0, & (\omega_1,\omega_2) \notin Q(0.6\pi) \end{cases}$$

where $\tau_1 = \tau_2 = 4$. The desired tolerance error ratio is $\rho_1/\rho_2 = 1$ and the FIR filter to be designed is a 9×9-point causal filter (i.e., $S = \{(n_1,n_2)|\ 0 \le n_1 \le 8,\ 0 \le n_2 \le 8\}$).

The parameters q, c and $\Delta\omega$ used were $q = 1.5, c = 10$ and $\Delta\omega = \pi/24$, respectively. The designed magnitude response and the absolute approximation error response are shown in Figures 2a and 2b, respectively. The absolute approximation error in the transition band is artificially

TABLE 1.

Numerical results of the designed zero-phase eightfold symmetric 5×5, 7×7 and 9×9-point lowpass filters by the proposed approximation method with $q = 1.5$ and $c = 10^{-4}$, and the corresponding results reported in [4] (Harris steepest ascent method) and those reported in [6] (the modified Lawson's method). The tolerance error ratio between the passband and the stopband is $\rho_1/\rho_2 = 1$.

Order	Algorithm	Maximum error	Iteration number
5×5	The proposed method	0.2718	6
	Harris' method	0.2670	—
	Modified Lawson algorithm	0.2733	7
7×7	The proposed method	0.1273	7
	Harris' method	0.1272	—
	Modified Lawson algorithm	0.1336	9
9×9	The proposed method	0.1189	5
	Harris' method	0.1142	—
	Modified Lawson algorithm	0.1202	8

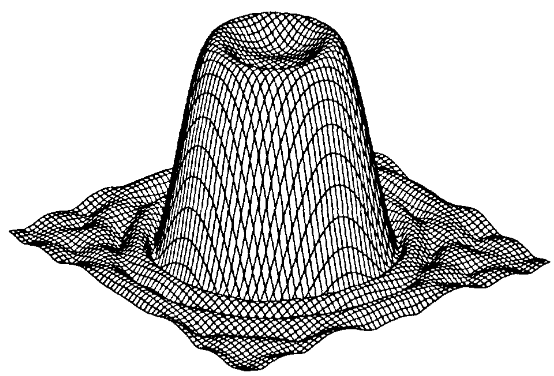

Fig. 1. Example 2: Magnitude response of the designed 10×10-point real causal nonlinear-phase lowpass filter with constant group delay.

set to zero, although it might not be very visible in Figure 2b. One can see, from Figure 2b, that basically there are two groups of local maxima of $|E(\omega_1, \omega_2)|$. One consists of smaller unequal local maxima of $|E(\omega_1, \omega_2)|$ at around the corners of the square region $\{(\omega_1, \omega_2)| - \pi \leq \omega_1 \leq \pi, -\pi \leq \omega_2 \leq \pi\}$. The other group consists of the rest of larger almost equal local maxima of $|E(\omega_1, \omega_2)|$. This is also consistent with the previous discussion of the 2-group distribution of final weighted-error local maxima mentioned in Section III. The number of iterations spent was 20. The maximum errors δ_1 in the passband and δ_2 in the stopband are 0.1166 and 0.1152, respectively. Note that $\delta_1/\delta_2 = 1.0126 \approx \rho_1/\rho_2 = 1$.

V. Conclusions

We have presented a new iterative WLS approximation method for the design of 2-D FIR filters. It can be used to design 2-D FIR filters with any arbitrary complex frequency response and the designed optimum filters are quasi-equiripple. We also showed three design examples to justify the good performance of the proposed approximation method no matter whether filter coefficients are real or complex. Moreover, the proposed approximation method does not have the degeneracy problem existing in exchange ascent algorithms because the WLS estimate (see (7)) is associated with an overdetermined linear vector model (see (5)).

Acknowledgments

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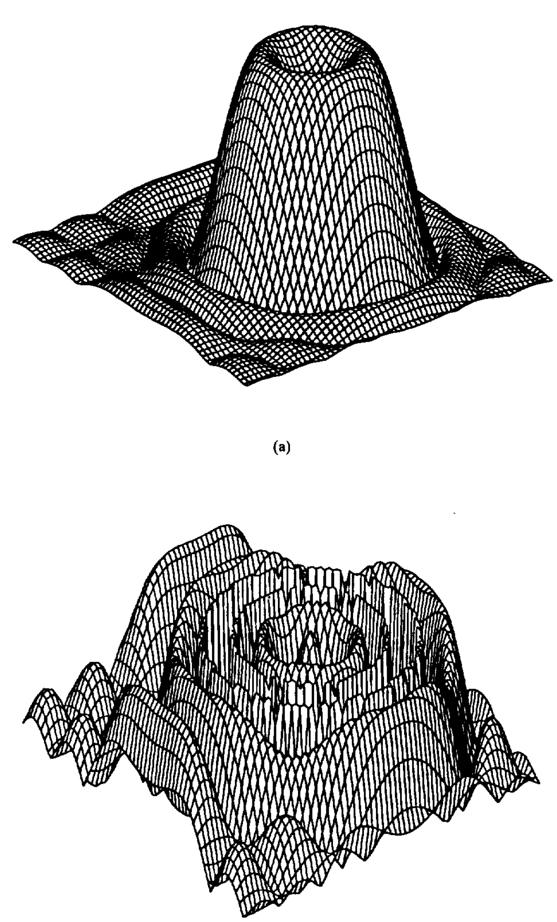

Fig. 2. Example 3: The designed 9×9-point causal circular-passband complex filter. (a) Magnitude response; (b) absolute approximation error response.